Advanced Algebra Exam (100 points), 2024-04-25

Throughout, $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{C}$.

27 $\frac{4}{11}$ 1. (30 points) Let $V = \mathbb{F}^{2\times 2}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$.

(1) Define the linear transformation $\phi : X \mapsto AX - XA$, prove that ϕ is not invertible.

-6, -4, -1. (2) Define the right multiplication transformation $A_{\mathrm{R}} : V \to V$ by $X \mapsto XA$, find the matrix of A_{R} under the basis $M = \{\boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}\}$, where:

$$\boldsymbol{E}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(3) For any $B \in V$, define the left multiplication transformation $B_{\mathrm{L}} : X \mapsto BX$ on V, prove that $A_{\mathrm{R}} B_{\mathrm{L}} = B_{\mathrm{L}} A_{\mathrm{R}}$.

18 $\frac{28}{33}$ 2. (20 points) Let ϕ be an invertible linear transformation on an $\mathbb{F}$-linear space V, and let W be a finite-dimensional ϕ-invariant subspace. Prove:

>10 (1) $\phi\,|_W$ is an invertible linear transformation on W.

(2) W is also an invariant subspace of ϕ^{-1} and $\phi^{-1}\,|_W = (\phi\,|_W)^{-1}$.

11 $\frac{17}{33}$ 3. (20 points) Let p be an odd prime number. Prove that

8, 12, 20.

$$g(y) = (p-1)y^{p-2} + (p-2)\mathrm{C}_p^1 y^{p-3} + \cdots + 2\mathrm{C}_p^{p-3} y + \mathrm{C}_p^{p-2}$$

and

$$f(x) = (p-1)x^{p-2} + (p-2)x^{p-3} + \cdots + 2x + 1$$

are irreducible over $\mathbb{Q}$. Here $\mathrm{C}_n^m = \frac{n!}{m!(n-m)!}$.

16 $\frac{14}{33}$ 4. (18 points) Let ψ and ϕ be two linear transformations on the $\mathbb{F}$-linear space V such that $\psi^2 = \psi$ and $\phi^2 = \phi$. Prove:

>12 (1) Im ψ = Im ϕ if and only if $\psi\phi = \phi$ and $\phi\psi = \psi$.

(2) Ker ψ = Ker ϕ if and only if $\psi\phi = \psi$ and $\phi\psi = \phi$.

5 $\frac{3}{11}$ 5. (12 points) Let V be an $\mathbb{F}$-linear space and $f, g_1, \cdots, g_s \in V^*$ such that

2.

$$\bigcap_{j=1}^{s} \mathrm{Ker}\, g_j \subseteq \mathrm{Ker}\, f.$$

Prove that there exist $b_1, b_2, \cdots, b_s \in \mathbb{F}$, such that

$$f = \sum_{j=1}^{s} b_j g_j.$$