20250227高代习题课

1. $a_1, a_2, \cdots, a_n$ are integers that are different from each other, let
$$f(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - 1.$$
(1) Prove that $f(x)$ is irreducible in the rational number field $\mathbb{Q}$.
(2) For integer $t \neq -1$, $g(x) = (x-a_1)\cdots(x-a_n)+t$.
Is $g(x)$ reducible in $\mathbb{Q}$?

2. $a_1, \cdots, a_n$ are integers that are different from each other. Prove:
$$f(x) = (x-a_1)\cdots(x-a_n)+1,$$
is irreducible in $\mathbb{Q}$, or it can be the square of rational coefficient polynomial.

3. Prove that $f(x) = (p-1)x^{p-2} + (p-2)x^{p-3} + \cdots + 2x+1$ is irreducible in $\mathbb{Q}$, where p is a prime number greater than 2.

4. p is a prime number, a is an integer. $f(x) = ax^p + px + 1$, and $p^2 \mid (a+1)$
Prove $f(x)$ is irreducible in $\mathbb{Q}$.

5. $f(x) \in \mathbb{Z}[x]$ (整系数多项式), x_1 and x_2 are different integers, and $f(x_1), f(x_2) = \pm 1$. Prove that
If $|x_1 - x_2| > 2$, $f(x)$ has no rational roots;
If $|x_1 - x_2| \le 2$, and $f(x)$ has rational roots, it must be $\frac{x_1+x_2}{2}$.

6. Prove that.

a non-zero complex number α is a root of a rational coefficient polynomial $\Leftrightarrow$ there is a rational coefficient polynomial $f(x)$, s.t. $f(\alpha)=\frac{1}{\alpha}$.

7. For an integer coefficient polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$,

it has integer roots $\Leftrightarrow$

there are $2(n-1)$ integers b_i, c_i such that

(1) $a_i = b_i + c_i$, $1 \leq i \leq n-1$

(2) $\frac{1}{c_1} = \frac{b_1}{c_2} = \frac{b_2}{c_3} = \cdots = \frac{b_{n-2}}{c_{n-1}} = \frac{b_{n-1}}{a_n}$.