# 0327 Tutorial

*Example* (1). Assume that $\sigma$ and $\tau$ are linear maps on an $n$-dimensional linear space $V$, and that $\sigma^2 = \tau^2 = \varepsilon$, where $\varepsilon$ is the identity transformation. Prove that:

Pf.

$$\mathrm{Im}(\sigma\tau - \tau\sigma) = \mathrm{Im}(\sigma + \tau) \cap \mathrm{Im}(\sigma - \tau).$$

Claim 1: $\mathrm{Im}(\sigma\tau - \tau\sigma) \subseteq \mathrm{Im}(\sigma+\tau) \cap \mathrm{Im}(\sigma-\tau)$

Observe: $v \in V$. $(\sigma+\tau)(\sigma-\tau)v = (\sigma+\tau)(\sigma v - \tau v)$

$= v - \sigma\tau v + \tau\sigma v - v = -(\sigma\tau - \tau\sigma)v.$

similarly $(\sigma-\tau)(\sigma+\tau)v = (\sigma-\tau)(\sigma v + \tau v)$

$= v + \sigma\tau v - \tau\sigma v - v = (\sigma\tau - \tau\sigma)v$

Indeed, otherwise, suppose $\exists\, \xi \in \mathrm{Im}(\sigma\tau - \tau\sigma)$. s.t.

$\xi \notin \mathrm{Im}(\sigma+\tau) \cap \mathrm{Im}(\sigma-\tau)$.

$\exists\, \xi_1 \in V$ s.t. $\xi = (\sigma\tau - \tau\sigma)\xi_1$.

But then $\exists$ $\xi_2 = (\sigma+\tau)\xi_1 \in V$ s.t. $\xi = (\sigma-\tau)\xi_2$

& $\xi_3 = (\sigma-\tau)\xi_1 \in V$ s.t. $\xi = (\sigma+\tau)\xi_3$.

therefore ※ to the assumption.

Claim 2: $\mathrm{Im}(\sigma\tau - \tau\sigma) \supseteq \mathrm{Im}(\sigma+\tau) \cap \mathrm{Im}(\sigma-\tau)$

Similarly, suppose $x \in \mathrm{Im}(\sigma+\tau) \cap \mathrm{Im}(\sigma-\tau)$

then $x \in \mathrm{Im}(\sigma+\tau)$ & $x \in \mathrm{Im}(\sigma-\tau)$.

then there is $x_1 \in V$ s.t. $x \in (\sigma+\tau)x_1$

$x_2 \in V$ s.t. $x \in (\sigma-\tau)x_2$.

Hence.

$(\sigma-\tau)x = (\sigma\tau - \tau\sigma)x_1$. ③

& $(\sigma+\tau)x = -(\sigma\tau - \tau\sigma)x_2$. ④

and so. $\frac{1}{2}$(③+④) $\Rightarrow$ $\sigma x = \frac{1}{2}(\sigma\tau - \tau\sigma)(x_1 - x_2)$

we note that. $v \in V$.

$\sigma(\sigma\tau - \tau\sigma)v = (\tau - \sigma\tau\sigma)v = (\tau\sigma^2 - \sigma\tau\sigma)v = (\tau\sigma - \sigma\tau)\sigma v.$

Therefore.

$x = \sigma^2 x = \frac{1}{2}(\sigma\tau - \tau\sigma)\sigma(x_1 - x_2)$ $\square$.

*Example* (2). Assume that $A$, $B : V \to V$ are linear maps, where $A$ is invertible, $B$ is nilpotent (i.e., $B^k = 0$ for some $k \in \mathbb{Z}_{>0}$), and $AB = BA$. Prove that $\ker(A - B) = 0$.

Pf: Take an arbitrary $\alpha \in \ker(A-B)$

$(A-B)\alpha = 0 \Rightarrow A\alpha = B\alpha$

Fix $k \in \mathbb{N}_+$. s.t. $B^k$ is the zero map. $B^k\alpha = 0$.

$B^{k-1}\alpha = A^{-1}AB^{k-1} = A^{-1}B^k = 0$.

$B^{k-2}\alpha = A^{-1}AB^{k-2} = A^{-1}B^{k-1} = 0$

. . . . .

$B\alpha = A^{-1}AB = A^{-1}B^2 = 0$.

$\alpha = A^{-1}A\alpha = A^{-1}B\alpha = 0$.

*Example* (3). Assume that $\sigma$ is a linear map on a linear space $V$, and that $\sigma^2 = \sigma$. Let $V_1 := \sigma(V)$ and $V_2 := \sigma^{-1}(0)$. Prove that $V = V_1 \oplus V_2$ and that for any $\alpha \in V_1$, $\sigma(\alpha) = \alpha$.

Notation: $\ker \sigma$.

Proof: (1) Clearly $V \supseteq V_1 + V_2$.

For any $a \in V$. we can write

$$\alpha = \sigma a + a - \sigma a.$$

let $a_1 := \sigma a$. & $a_2 := a - \sigma a$.

then clearly $a_1 \in V_1$.

and $\sigma a_2 = \sigma a - \sigma^2 a = 0$. so $c \in V_2$.

Hence. $a = a_1 + a_2 \in V_1 + V_2$.

and so $V = V_1 + V_2$.

Moreover, take an arbitrary $b \in V_1 \cap V_2$.

then $\sigma b = 0$. (as $b \in V_2$)

& $\exists\, b' \in V$ s.t.

(as $b \in V_1$). $b = \sigma b' = \sigma^2 b' = \sigma(\sigma b') = \sigma b = 0$

(2) For any $c \in V_1$. there $\exists\, c' \in V$ s.t. $\sigma c' = c$

& hence $c = \sigma c' = \sigma^2 c' = \sigma(\sigma c') = \sigma c$ $\square$.

*Example* (5). $\varphi$ is a linear map on an $n$-dimensional linear space $V$. Prove that the following claims are equivalent:

$$(1)\quad V = \ker\varphi \oplus \mathrm{Im}\varphi; \qquad (2)\quad \ker\varphi \cap \mathrm{Im}\varphi = \{0\};$$
$$(3)\quad \ker\varphi = \ker\varphi^2; \qquad (4)\quad \mathrm{Im}\varphi = \mathrm{Im}\varphi^2.$$

Pf: (2) ⇒ (1) It is enough for us to show that
$V = \ker\varphi + \mathrm{Im}\,\varphi$.
then, the claim follows from the def.s
clearly $\ker\varphi \subseteq V$. & $\mathrm{Im}\,\varphi \subseteq V \Rightarrow \ker\varphi + \mathrm{Im}\varphi \subseteq V$.
and $\dim V = \dim\ker\varphi + \dim\mathrm{Im}\varphi$. (Fun. thm of Lin. alg.).
So. $V = \ker\varphi + \mathrm{Im}\,\varphi$.

(1) ⇒ (4)
I. take arbitrary $a \in \mathrm{Im}\,\varphi^2$. then $\exists\, b \in V$. s.t.
$a = \varphi^2 b = \varphi(\varphi b)$. $\mathrm{Im}\,\varphi^2 \subseteq \mathrm{Im}\,\varphi$.

II. take arbitrary $c \in \mathrm{Im}\,\varphi$. $\exists\, d \in V$ s.t. $c = \varphi d$
moreover. from (1). there is a decomposition. that.
$d = d_1 + d_2$ where $d_1 \in \ker\varphi$ & $d_2 \in \mathrm{Im}\varphi$.
and there is $d_3 \in V$ s.t. $d_2 = \varphi d_3$.
Therefore $c = \varphi d = \varphi d_2 = \varphi^2 d_3$.
Hence I & II combine to imply the claim.

(4) ⇒ (3). $\ker\varphi \subseteq \ker\varphi^2$ is clear.
$\dim(\ker\varphi) = \dim V - \dim\mathrm{Im}\,\varphi^2 = \dim\ker\varphi^2$.
So $\ker\varphi = \ker\varphi^2$.

(3) ⇒ (2) Let $s \in \mathrm{Im}\,\varphi \cap \ker\varphi$.
there is $t \in V$. $s = \varphi t$. as $s \in \mathrm{Im}\,\varphi$.
and $\varphi s = \varphi^2 t = 0$. as $s \in \ker\varphi$.
this implies that $t \in \ker\varphi^2 = \ker\varphi$
Hence $s = \varphi t = 0$. ∎

$\mathrm{Im}\, A \subseteq \ker A.$

$\mathrm{Im}\, B \subseteq \ker B.$

*Example* (6). Assume that $V$ is a finite dimensional linear space over a number field $\mathbb{F}$. Let $A$ and $B$ be two linear maps on $V$ such that $A^2 = B^2 = 0$ and $AB + BA = I$, where $I$ is the identity map. Prove that

(1) $\ker A = A(\ker B)$, $\ker B = B(\ker A)$, $V = \ker A \oplus \ker B$.

(2) $\dim V$ is even.

Pf:

1) ① $\ker A \subseteq A(\ker B)$.

For any $\alpha \in \ker A \subseteq V$, one has

WTS $\alpha$ in form of $A$ & $B$.

$\alpha = (AB+BA)\alpha = AB\alpha + B\overbrace{A\alpha}^{0} = AB\alpha.$

moreover $B^2\alpha = B(B\alpha) = 0 \Rightarrow B\alpha \in \ker B$.

Hence $\alpha \in A(\ker B)$.

$\ker A \supseteq A(\ker B)$.

Take an arbitrary $\beta \in A(\ker B)$.

then $\exists\, \beta' \in \ker B$ s.t. $\beta = A\beta'$ WTS $\beta \in \ker A$.

$\mathrm{Im}\, A \subseteq \ker A$.

$A\beta = A^2\beta' = 0$

② $\ker B \subseteq B(\ker A)$. Exer.

③ For any $\gamma \in V$, we can write

$\gamma = AB\gamma + BA\gamma$ WTS: $V = \ker A + \ker B$.

$\ker A = A\ker B \supseteq A\,\mathrm{Im}\, B$, $\ker B = B\ker A \supseteq B\,\mathrm{Im}\, A$.

From the proof of claim ①, we know

$B\gamma \in \ker B$, and so $AB\gamma \subseteq \ker A$.

$A\gamma \in \ker A$, and so $BA\gamma \subseteq \ker B$.

Hence $V = \ker A + \ker B$.

Finally, for any $v \in \ker A \cap \ker B \subseteq V$,

then $v = ABv + BAv = 0$.

Therefore $V = \ker A \oplus \ker B$.

2) From claim (1)③, we have $\dim V = \dim \ker A + \dim \ker B$. ①

We note that $\ker A = A \underbrace{\ker B}_{} \subseteq \mathrm{Im}\, A$

and so $\dim \ker A \leq \dim \mathrm{Im}\, A = n - \dim \ker A \Rightarrow \dim \ker A \leq \frac{n}{2}$ ②

Similarly $\ker B \leq \frac{n}{2}$.

moreover ① & ② combine to imply that $\dim \ker B = \frac{n}{2}$.

Finally, we recall that from definition $\dim \ker B = r \in \mathbb{N}$.

Hence $n = 2r$. $\square$

Alternatively. $\mathrm{Im}\, A \subseteq \ker A$.

$\ker A \subseteq \mathrm{Im}\, A$.

$\Rightarrow \ker A = \mathrm{Im}\, A$.

then the result follows from Fun. thm of lin. alg.

i.e. Rank-Nullity thm.

Pf:

*Example* (7). Assume $\varphi$ is a linear map on an $n$-dimensional linear space $V$. Prove that there exists a natrual number $r$ such that:

$$(1) \quad \ker\varphi^r = \ker\varphi^{r+1}; \quad (2) \quad \text{for any natrual number } s, \ker\varphi^r = \ker\varphi^{r+s}.$$

(1). Clearly,

$\ker\varphi \subseteq \ker\varphi^2 \subseteq \cdots \subseteq \ker\varphi^k \subseteq \ker\varphi^{k+1} \subseteq \cdots$

and so

$\dim\ker\varphi \leq \dim\ker\varphi^2 \leq \cdots \leq \dim\ker\varphi^k \leq \cdots$

However, for $\forall i \in \mathbb{Z}_+$

$0 \leq \dim\ker\varphi^i \leq n$. has finite length.

Hence there must exist $r \in \mathbb{Z}_+$ s.t.

$\dim\ker\varphi^r = \dim\ker\varphi^{r+1}$.

(2) Prove by mathematical induction.

① Base case: $s=1$. True follows from claim (1).

② Induction Step:

WTS: $\ker\varphi^r = \ker\varphi^{r+s-1} \Rightarrow \ker\varphi^r = \ker\varphi^{r+s}$

Induction Hypothesis: $\ker\varphi^r = \ker\varphi^{r+s-1}$.

We note that $\ker\varphi^r \subseteq \ker\varphi^{r+s}$ straightforward.

It is enough to show that $\ker\varphi^r \supseteq \ker\varphi^{r+s}$.

To do this, take an arbitrary $\alpha \in \ker\varphi^{r+s}$.

$0 = \varphi^{r+s}\alpha = \varphi^{r+s-1}(\varphi\alpha)$

that is $\varphi\alpha \in \ker\varphi^{r+s-1} = \ker\varphi^r$

hence $0 = \varphi^r(\varphi\alpha) = \varphi^{r+1}\alpha$.

$\alpha \in \ker\varphi^{r+1} = \ker\varphi$ from base case.

$\square$