Homework 6

March 29, 2025

1. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be a basis of a linear space V over the number field $\mathbb{F}$, and f_1, f_2, f_3 be the dual basis. Let $\alpha_1 = \epsilon_1 + \epsilon_2 + \epsilon_3, \alpha_2 = \epsilon_2 + \epsilon_3, \alpha_3 = \epsilon_3$.
   (a) Prove that $\alpha_1, \alpha_2, \alpha_3$ form a basis of V.
   (b) Please find the dual basis of $\alpha_1, \alpha_2, \alpha_3$ in terms of f_1, f_2, f_3.
2. Fix $a \in \mathbb{R}$. Let $\mathscr{A}(f(x)) = f(x+a) - f(x), \forall f(x) \in \mathbb{R}[x]_n$. Prove that $\mathscr{A}$ is a linear transformation on $\mathbb{R}[x]$, and please find a matrix of $\mathscr{A}$ under the basis $1, x-a, \frac{1}{2!}(x-a)^2, \cdots, \frac{1}{(n-1)!}(x-a)^{n-1}$.
3. Let V and W be two linear spaces of dimension n and m over $\mathbb{R}$, and φ is a linear transformation from V to W. Please prove that there exist bases of V and W such that the matrix of φ under the given bases is
$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad r = \text{rank}(\varphi)$$
4. Let A be an $n \times n$ matrix over $\mathbb{R}$. Let φ be a linear transformation on the linear space of all $n \times n$ matrices over $\mathbb{R}$ $M_{n\times n}(\mathbb{R})$ defined by $\varphi(X) = AX, \forall X \in M_{n\times n}(\mathbb{R})$. Please find the trace and determinant of the map φ.
5. Let $\varphi : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be a linear map. Please prove that :
   - There is a unique $C \in M_{n\times n}(\mathbb{R})$, such that $\varphi(A) = \text{trace}(AC), \forall A \in M_{n\times n}(\mathbb{R})$;
   - If $\forall A, B \in M_{n\times n}(\mathbb{R}), \varphi(AB) = \varphi(BA)$, then there is a $\lambda \in \mathbb{R}$, such that $\varphi(A) = \lambda \text{trace}(A), \forall A \in M_{n\times n}(\mathbb{R})$.
6. Let V be a linear space of dimension n. Please determine whether there are linear transformations σ, τ on V such that $\sigma\tau - \tau\sigma = I$? What about when the V is of infinite dimension?
7. Let V be an n dimensional linear space over $\mathbb{F}$, $\varphi_1, \cdots, \varphi_s$ be linear transformations on V, and $\varphi = \sum_{i=1}^{s} \varphi_i$. Prove that φ is unipotent $\varphi^2 = \varphi$ and $\text{rank}(\varphi) = \sum_{i=1}^{s} \text{rank}\varphi_i$ if and only if φ_i is unipotent $\varphi_i^2 = \varphi_i$ for $i = 1, \cdots, s$ and $\varphi_i\varphi_j = 0$ for $i \neq j$.
8. Let $\varphi : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be a linear transformation defined by $\varphi(X) = AXA^T$ for $X \in M_{n\times n}(\mathbb{R})$, where A is an $n \times n$ matrix over $\mathbb{R}$. Please find the dimension of $im(\varphi)$ and a basis.